

# Isospin-breaking corrections to the pion–nucleon scattering lengths

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We analyze isospin breaking through quark mass differences and virtual photons in the pion–nucleon scattering lengths in all physical channels in the framework of covariant baryon chiral perturbation theory. The so-called triangle relation is found to be violated by about 1.5 %. We encounter a substantial isospin-breaking correction to neutral-pion–nucleon scattering beyond Weinberg’s prediction due to a cusp effect. Finally, the application to hadronic atoms is briefly discussed.

*6th International Workshop on Chiral Dynamics*  
*July 6-10 2009*  
*Bern, Switzerland*

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## 1. Introduction

Isospin violation in the Standard Model is driven by strong and electromagnetic interactions, that is by the differences in the light quark masses and charges, respectively. Already in [1] Weinberg stressed that the pion–nucleon scattering lengths offer a particularly good testing ground for strong isospin violation, predicting e.g. a large effect in the difference of the neutral-pion–nucleon scattering lengths  $a_{\pi^0 p} - a_{\pi^0 n}$ . Isospin violation in  $\pi N$  scattering was addressed in the framework of heavy-baryon chiral perturbation theory (ChPT) in a series of papers about a decade ago [2–6]. Recently, new interest arose in high-precision calculations of the pion–nucleon scattering lengths. First, corrections for isospin violation are an essential ingredient to extract the isoscalar and isovector  $\pi N$  scattering lengths  $a^+$  and  $a^-$  from pionic hydrogen ( $\pi H$ ) and deuterium ( $\pi D$ ) measurements to high precision. The isospin-breaking corrections to  $a_{\pi^- p \rightarrow \pi^- p}$  needed for the ground state of  $\pi H$  were calculated in [7] at third chiral order. A consistent description requires the knowledge of  $a_{\pi^- p \rightarrow \pi^0 n}$  (width of  $\pi H$ ) and  $a_{\pi^- n \rightarrow \pi^- n}$  (ground state of  $\pi D$ ) at the same accuracy. In the analysis of  $\pi D$  isospin violation is particularly important, since the  $\pi d$  scattering length at leading order is proportional to the small isoscalar scattering length  $a^+$  and therefore chirally suppressed [8]. Second, threshold pion photoproduction offers the unique possibility of measuring the so far undetermined  $\pi^0 p$  scattering length and gives access to the charge exchange scattering length  $a_{\pi^+ n \rightarrow \pi^0 p}$  [9, 10]. Such measurements are becoming feasible at HIγS and at MAMI. In view of these developments, we have extended the work of [7] to *all* charge channels in pion–nucleon scattering in [11].

## 2. Formalism

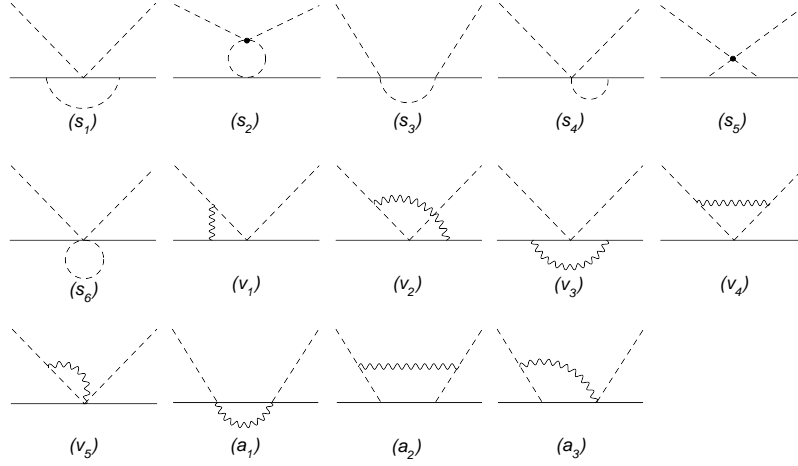
The calculation of the scattering lengths is performed at  $\mathcal{O}(p^3)$  in manifestly covariant baryon ChPT [12] and at first order in the isospin-breaking parameter  $\delta = \mathcal{O}(e^2, m_d - m_u)$ , using the effective Lagrangian for nucleons, pions, and virtual photons, as constructed in [7]. We denote the masses of proton, neutron, charged and neutral pion by  $m_p$ ,  $m_n$ ,  $M_\pi$ , and  $M_{\pi^0}$ , respectively, and define the isospin limit by the charged particles. The mass differences are expressed by  $\Delta_\pi = M_\pi^2 - M_{\pi^0}^2$  and  $\Delta_N = m_n - m_p$ .

The one-loop topologies are depicted in Fig. 1. As soon as we take into account virtual photons, we have to specify how to deal with the threshold divergences: first of all, we subtract all one-photon-reducible diagrams, since they diverge  $\sim 1/t$  ( $s$  and  $t$  are the usual Mandelstam variables), and denote the resulting amplitude by  $\tilde{T}$ . The additional divergences due to photon loops may be regularized in the form

$$e^{iQ\alpha\theta_C(|\mathbf{p}|)} \tilde{T} \Big|_{|\mathbf{p}| \rightarrow 0} = \frac{\beta_1}{|\mathbf{p}|} + \beta_2 \log \frac{|\mathbf{p}|}{\mu_c} + T_{\text{thr}} + \mathcal{O}(|\mathbf{p}|), \quad (2.1)$$

where  $\mathbf{p}$  denotes the center-of-mass momentum,  $\alpha = e^2/4\pi$  the fine structure constant,  $\theta_C(|\mathbf{p}|)$  the infrared divergent Coulomb phase given by

$$\theta_C(|\mathbf{p}|) = -\frac{\mu_c}{|\mathbf{p}|} \log \frac{m_\gamma}{2|\mathbf{p}|}, \quad (2.2)$$



**Figure 1:** One-loop topologies for  $\pi N$  scattering at threshold. Solid, dashed, and wiggly lines, denote nucleons, pions, and photons, respectively. Crossed diagrams and diagrams contributing via wave function renormalization only are not shown.

$\mu_c = m_p M_\pi / (m_p + M_\pi)$  the reduced mass of the incoming particles, and  $Q$  accounts for the charges of the particles involved. The term  $\propto 1/|\mathbf{p}|$  referred to as Coulomb pole is solely generated by  $(v_1)$  at this order, while  $\beta_2$  only enters at two-loop level. The scattering lengths are finally given by

$$a = \frac{T_{\text{thr}}}{8\pi\sqrt{s}}. \quad (2.3)$$

It turns out that for the charged-pion elastic channels the triangle graph  $(s_5)$  yields a very large contribution. In addition,  $(s_3)$  can generate cusp effects, which are proportional to  $\sqrt{\delta}$  and thus enhanced by  $\sqrt{\delta}$ . For the analytical results and more details of the calculation we refer to [11].

### 3. Numerical results

In the isospin limit, the  $\pi N$  scattering lengths are solely determined by  $a^+$  and  $a^-$ . Subtracting these contributions, we obtain the following isospin-breaking shifts (in units of  $10^{-3} M_\pi^{-1}$ ):

isospin limit	channel	shift	channel	shift
$a^+ + a^-$	$\pi^- p \rightarrow \pi^- p$	$-3.4^{+4.3}_{-6.5} + 5.0i$	$\pi^+ n \rightarrow \pi^+ n$	$-4.3^{+4.3}_{-6.5} + 6.0i$
$a^+ - a^-$	$\pi^+ p \rightarrow \pi^+ p$	$-5.3^{+4.3}_{-6.5}$	$\pi^- n \rightarrow \pi^- n$	$-6.2^{+4.3}_{-6.5}$
$-\sqrt{2}a^-$	$\pi^- p \rightarrow \pi^0 n$	$0.4 \pm 0.9$	$\pi^+ n \rightarrow \pi^0 p$	$2.3 \pm 0.9$
$a^+$	$\pi^0 p \rightarrow \pi^0 p$	$-5.2 \pm 0.2$	$\pi^0 n \rightarrow \pi^0 n$	$-1.8 \pm 0.2$

The precise values of the low-energy constants as well as a detailed estimate of the theoretical uncertainties can be found in [11]. Our result for the triangle relation, which vanishes in the isospin limit and is thus a convenient way to quantify isospin violation in terms of measurable quantities, reads

$$R = 2 \frac{a_{\pi^+ p \rightarrow \pi^+ p} - a_{\pi^- p \rightarrow \pi^- p} - \sqrt{2} a_{\pi^- p \rightarrow \pi^0 n}}{a_{\pi^+ p \rightarrow \pi^+ p} - a_{\pi^- p \rightarrow \pi^- p} + \sqrt{2} a_{\pi^- p \rightarrow \pi^0 n}} = (1.5 \pm 1.1) \%. \quad (3.1)$$

The difference between the elastic neutral-pion–nucleon scattering lengths is found to be

$$\begin{aligned} a_{\pi^0 p} - a_{\pi^0 n} &= \frac{m_p}{4\pi(m_p + M_\pi)} \left\{ \frac{4c_5 B(m_d - m_u)}{F_\pi^2} - \frac{M_\pi^2}{8\pi F_\pi^4} \left( \sqrt{\Delta_\pi + 2M_\pi \Delta_N} - \sqrt{\Delta_\pi - 2M_\pi \Delta_N} \right) \right\} \\ &= ((-2.3 \pm 0.4) - 1.1) \cdot 10^{-3} M_\pi^{-1} = (-3.4 \pm 0.4) \cdot 10^{-3} M_\pi^{-1}. \end{aligned} \quad (3.2)$$

The first term was already given by Weinberg in [1], while the second one is due to a cusp effect and contributes roughly one third to the final result.

#### 4. Application to hadronic atoms

$a^+$  and  $a^-$  can be related to the strong shift of the ground state of  $\pi H$  and  $\pi D$  and to the width of  $\pi H$  via Deser-type formulae [13–16], to which isospin-breaking corrections are an essential ingredient. The impact of our isospin-breaking corrections on the extraction of  $a^-$  and

$$\tilde{a}^+ = a^+ + \frac{m_p}{4\pi(m_p + M_\pi)} \left\{ \frac{4\Delta_\pi}{F_\pi^2} c_1 - 2e^2 f_1 \right\} \quad (4.1)$$

is displayed in Fig. 2:<sup>1</sup> the constraint due the width of  $\pi H$  barely changes, while the the bands for the level shift of  $\pi H$  and  $\pi D$  significantly move upwards when going from  $\mathcal{O}(p^2)$  to  $\mathcal{O}(p^3)$ , corresponding to the small corrections to the charge exchange reaction and a large shift in the charged-pion elastic channels due to the triangle graph alluded to above.

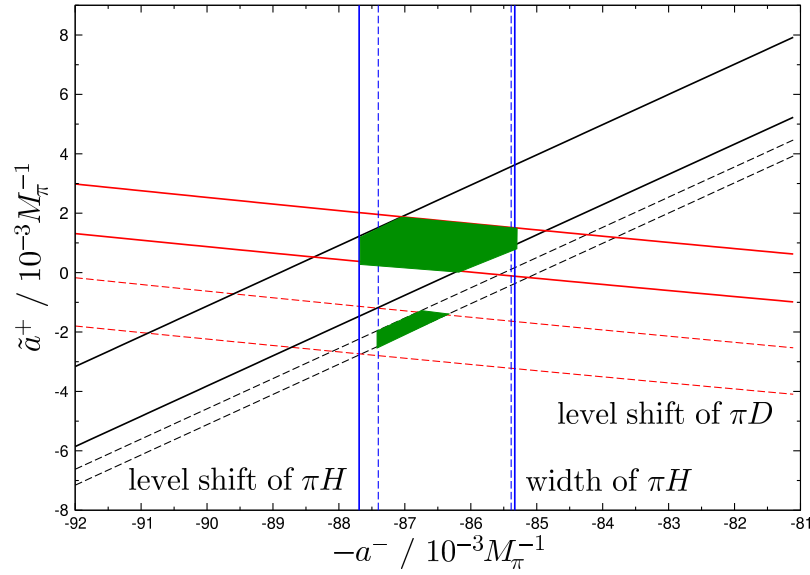
However, Fig. 2 does not provide a complete picture at  $\mathcal{O}(p^3)$ : the few-body contributions to  $a_{\pi d}$  [20–22] are still based on the assumption of isospin symmetry. It remains to be seen whether the nice consistency between the three bands persists once isospin violation in the few-body part is included.

#### 5. Summary and outlook

We have systematically analyzed isospin violation in the  $\pi N$  scattering lengths in all channels, including an estimate of the theoretical uncertainties. We find that isospin breaking is quite small in  $\pi^- p \rightarrow \pi^0 n$ , at the order of one percent at most, whereas the charged-pion elastic channels display more sizeable effects on the few-percent level. In particular, the triangle relation is violated by about 1.5% consistent with earlier findings in heavy-baryon ChPT and inconsistent with the 5–7% deviation extracted from the data at lowest pion momenta in [23, 24]. In addition, we find a substantial isospin-breaking correction to the neutral-pion–proton scattering length. Finally, we have shown that in a full  $\mathcal{O}(p^3)$  calculation the value of the isoscalar scattering length will increase as compared to previous analyses of hadronic atoms [8].

An extension of this analysis beyond threshold for  $\pi^\pm p \rightarrow \pi^\pm p$  and  $\pi^- p \rightarrow \pi^0 n$  can be found in [25], while a full calculation of isospin violation in the few-body contributions to the  $\pi d$  scattering length will be addressed in [26].

<sup>1</sup>Experimental input: level shift of  $\pi H$ :  $\varepsilon_{1s} = (-7.120 \pm 0.017) \text{ eV}$  [17], width of  $\pi H$ :  $\Gamma_{1s} = (0.823 \pm 0.019) \text{ eV}$  [18], and  $\pi^- d$  scattering length:  $a_{\pi d} = (-0.0261 \pm 0.0005) M_\pi^{-1}$  [19].



**Figure 2:** Constraints on  $a^-$  and  $\tilde{a}^+$  provided by the level shift of  $\pi H$  and  $\pi D$  and the width of  $\pi H$ . Solid (dashed) bands refer to isospin-breaking corrections at  $\mathcal{O}(p^3)$  ( $\mathcal{O}(p^2)$ ), respectively.

## Acknowledgments

Partial financial support by the Helmholtz Association through funds provided to the virtual institute “Spin and strong QCD” (VH-VI-231), by the European Community-Research Infrastructure Integrating Activity “Study of Strongly Interacting Matter” (acronym HadronPhysics2, Grant Agreement n. 227431) under the Seventh Framework Programme of the EU, by DFG (SFB/TR 16, “Subnuclear Structure of Matter”) and by the Bonn-Cologne Graduate School of Physics and Astronomy is gratefully acknowledged.

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